HEISENBERG-EULER EFFECTIVE LAGRANGIAN

INTRODUCCIÓN A LA TEORÍA CUÁNTICA DE CAMPOS I

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1. Effective Lagrangians

Effective Lagrangians incorporate to classical theory corrections which are induced by quantum effects, like vacuum polarization. Usually, this gives rise to non-renormalizable Lagrangians.

The paradigm of these procedures are the seminal papers by Heisenberg-Euler and by Weisskopf (and some years later by Schwinger), which incorporate non-linear corrections to the Maxwell Lagrangian for the (low energy - large wavelength) electromagnetic background field, due to its interaction with the Dirac electron field:

- W. Heinsenberg and H. Euler, *Consequences of Dirac's Theory of Positrons*, Z. Phys. 98 (1936) 714.
- V. Weisskopf, *The electrodynamics of the vacuum based on the quantum theory of electrons*, English translation in Early Quantum Electrodynamics: A source book, A.I. Miller Edt. Cambridge University Press (1994).
- J. Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. **82** (1951) 664.

2. The *proper time* approach

In spinor QED, the *vacuum-to-vacuum amplitud* for the Dirac field in the presence of a *background* (classical) electromagnetic field A_μ is given in terms of **functional determinant**, a *formal (gauge-invariant) expression* which admits the following representation,

$$S_0[A] := \langle \Omega_0 \, | S[A] | \, \Omega_0 \rangle =$$

$$(2.1)\qquad \begin{aligned} &= \operatorname{Det}\left\{ \left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m + i\varepsilon\right] \left[\gamma^\mu i\partial_\mu - m + i\varepsilon\right]^{-1} \right\} \\ &= \exp \operatorname{Tr} \log \left\{ \left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m + i\varepsilon\right] \left[\gamma^\mu i\partial_\mu - m + i\varepsilon\right]^{-1} \right\} \\ &= \exp \operatorname{Tr} \log \left\{ \left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) + m - i\varepsilon\right] \left[\gamma^\mu i\partial_\mu + m - i\varepsilon\right]^{-1} \right\} \end{aligned}$$

Date: June 21, 2022.

where, in the last step, we have taken into account that, with $\mathcal{C} = i\gamma^0\gamma^2$, $\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -(\gamma^\mu)^t$ and we have assumed that (at least formally) the trace is left invariant by cyclic permutations. For definiteness, in the following we take $e > 0$.

So, we can write

$$
\begin{gathered}
2\log S_0[A] = \\
\operatorname{Tr}\log\left\{\left[\left(\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\right)^2 - (m^2 - i\varepsilon)\right]\left[\left(\gamma^\mu i\partial_\mu\right)^2 - (m^2 - i\varepsilon)\right]^{-1}\right\}
\end{gathered} \tag{2.2}
$$

and, taking into account the well known relation

$$
\log\left(\frac{a}{b}\right) = -\int_0^\infty \frac{ds}{s}\left\{e^{is(a+i\varepsilon)} - e^{is(b+i\varepsilon)}\right\} \tag{2.3}
$$

for $\varepsilon > 0$, we finally get the so-called *proper time representation*,

$$
\begin{gathered}
2\log S_0[A] = \\
-\operatorname{Tr}\int_0^\infty \frac{ds}{s}\, e^{is\left(-m^2 + i\varepsilon\right)}\left\{e^{is\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\right]^2} - e^{is\left[\gamma^\mu i\partial_\mu\right]^2}\right\}.
\end{gathered} \tag{2.4}
$$

Now, we have

$$
\begin{gathered}
\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\right]^2 = \\
\left\{\tfrac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\} + \tfrac{1}{2}\left[\gamma^\mu,\gamma^\nu\right]\right\}\left(i\partial_\mu - eA_\mu\right)\left(i\partial_\nu - eA_\nu\right) = \\
= \left\{\eta^{\mu\nu}\mathbf{1}_4 - i\sigma^{\mu\nu}\right\}\left(i\partial_\mu - eA_\mu\right)\left(i\partial_\nu - eA_\nu\right) \\
= \left(i\partial - eA\right)^2\mathbf{1}_4 - \tfrac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\,,
\end{gathered} \tag{2.5}
$$

where $\sigma^{\mu\nu} = \frac{\imath}{2}\left[\gamma^\mu,\gamma^\nu\right]$.

It is not possible to evaluate the RHS of (2.4) for an arbitrary configuration of the background field. But we can go further by taking constant and uniform (slowly varying) electromagnetic intensities, since in this case we can write.

$$
e^{is\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\right]^2} = e^{is\left(i\partial - eA\right)^2} e^{is\left(-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)}. \tag{2.6}
$$

Notice that there is a scalar and a pseudo-scalar which can be constructed from $F_{\mu\nu}$,

$$
\mathcal{F} := \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad \mathcal{G} := \mathbf{E}\cdot\mathbf{B} = -\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu}\,. \tag{2.7}
$$

If both invariants are non-vanishing, one can always adopt a reference frame in which **E** and **B** are parallel and point in the third axis direction. We take $F_{03} = E > 0$, $F_{12} = B$.

A suitable vector potential describing this situation is given by

$$A_0 = -Ex^3\,, \quad A_1 = 0\,, \quad A_2 = Bx^1\,, \quad A_3 = 0\,. \tag{2.8}$$

In this case we get

$$\begin{aligned} e^{-is\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}} &= e^{-ise\left(\sigma^{03}E+\sigma^{12}B\right)} = \\ &= e^{-iseE\sigma^{03}}e^{-iseB\sigma^{12}}\,, \end{aligned} \tag{2.9}$$

since $[\sigma^{03}, \sigma^{12}] = -[\gamma^0\gamma^3, \gamma^1\gamma^2] = 0$.

On the other hand,

$$(i\partial - eA)^2 = \left(i\partial_0 + eEx^3\right)^2 - (i\partial_1)^2 - \left(i\partial_2 - eBx^1\right)^2 - (i\partial_3)^2\ . \tag{2.10}$$

This can be written as

$$\begin{aligned} (i\partial - eA)^2 &= -\left\{{P_1}^2 + (eB)^2\left(x^1 + \tfrac{P_2}{eB}\right)^2\right\} - \\ &\quad -\left\{P_3^2 + (-ieE)^2\left(x^3 - \tfrac{P_0}{eE}\right)^2\right\} \\ &= -e^{i\left(\frac{P_2}{eB}\right)P_1}\left\{{P_1}^2 + |eB|^2(x^1)^2\right\}e^{-i\left(\frac{P_2}{eB}\right)P_1} - \\ &\quad -e^{i\left(-\frac{P_0}{eE}\right)P_3}\left\{{P_3}^2 + (-ieE)^2\,(x^3)^2\right\}e^{-i\left(-\frac{P_0}{eE}\right)P_3}\,, \end{aligned} \tag{2.11}$$

where $P_\mu = -i\partial_\mu$ are the *translation operators* in the direction of x^μ.

We recognize in the first brackets in the RHS of this equation the Hamiltonian of a harmonic oscillator of mass $m = 1/2$ and frequency $\omega = |2eB|$ and in the second one the Hamiltonian of another harmonic oscillator of mass $m = 1/2$ and (complex) frequency $\omega = -i2eE(1+\imath 0)$ (with a small positive real part).

So, we can write

(2.12)

$$
\begin{aligned}
&2\log S_0[A] = \\
&-\int d^4x \int_0^\infty \frac{ds}{s}\, e^{is\left(-m^2+i\varepsilon\right)} \left\{ \mathrm{tr}\left[e^{-iseE\sigma^{03}} e^{-iseB\sigma^{12}} \right] \right. \\
&\times \langle \mathbf{x}|\, e^{i\left(\frac{P_2P_1}{eB}-\frac{P_0P_3}{eE}\right)} e^{-is\left\{ {P_1}^2+|eB|^2\left(x^1\right)^2 \right\}} \\
&e^{-is\left\{ P_3^2+(-ieE)^2\left(x^3\right)^2 \right\}} e^{-i\left(\frac{P_2P_1}{eB}-\frac{P_0P_3}{eE}\right)} |\mathbf{x}\rangle - \\
&\left. -4\langle \mathbf{x}|\, e^{is\left(\mathbf{P}\right)^2} |\mathbf{x}\rangle \right\} .
\end{aligned}
$$

Now, inserting the identity in the form $\int d^4p\, |\mathbf{p}\rangle\langle\mathbf{p}|$, and taking into account that

$$
\langle \mathbf{x}|\mathbf{p}\rangle = \frac{e^{ix^\mu p_\mu}}{(2\pi)^2}\,, \tag{2.13}
$$

we get for the first matrix element in the brackets

(2.14)

$$
\begin{aligned}
M = \frac{1}{(2\pi)^4} \int d^4p \int d^4p'\ e^{ix^\mu\left(p_\mu - p'_\mu\right)} \\
\times e^{i\left(\frac{p_2p_1}{eB}-\frac{p_0p_3}{eE}\right)} e^{-i\left(\frac{p'_2p'_1}{eB}-\frac{p'_0p'_3}{eE}\right)} \delta\left(p_0-p'_0\right)\delta\left(p_2-p'_2\right)\times \\
\langle p_1| e^{-is\left\{ {P_1}^2+|eB|^2\left(x^1\right)^2 \right\}} |p_1\rangle \langle p_3| e^{-is\left\{ P_3^2+\left[-ieE(1+\imath 0)\right]^2\left(x^3\right)^2 \right\}} |p_3\rangle
\end{aligned}
$$

And since

$$
\begin{aligned}
\frac{1}{(2\pi)^2} \int dp_0 \int dp_2\ e^{-i\frac{p_0}{eE}\left(p_3-p'_3\right)} e^{i\frac{p_2}{eB}\left(p_1-p'_1\right)} = \\
= e^2E|B|\,\delta\left(p_1-p'_1\right)\delta\left(p_3-p'_3\right)\,,
\end{aligned} \tag{2.15}
$$

M does not depends on x and reduces to

$$M = \frac{e^2 E|B|}{(2\pi)^2} \int dp_1 \, \langle p_1| e^{-is\left\{{P_1}^2 + |eB|^2 \left(x^1\right)^2\right\}} |p_1\rangle \tag{2.16}$$
$$\times \int dp_3 \, \langle p_3| e^{-is\left\{P_3^2 + [-ieE(1+\imath 0)]^2 \left(x^3\right)^2\right\}} |p_3\rangle \, ,$$

where the remaining integrals give the traces of the evolution operators of the two harmonic oscillators in the *proper time* s.

Taking into account that, for the Hamiltonian $H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}X^2$, we have

$$\mathrm{Tr}\left\{e^{-\beta H}\right\} = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)} \tag{2.17}$$

for $\Re(\beta\omega) > 0$, we get

$$M = \frac{1}{(2\pi)^2} \frac{|eB|}{2\sinh\left(is|eB|\right)} \; \frac{eE}{2\sinh\left(is\left[-ieE(1+\imath 0)\right]\right)} = \tag{2.18}$$
$$= -\frac{i}{16\pi^2} \, \frac{|eB|}{\sin\left(s|eB|\right)} \; \frac{eE}{\sinh\left(seE(1+\imath 0)\right)} \, .$$

On the other hand, taking into account that $(i\sigma^{03})^2 = -(i\gamma^0\gamma^3)^2 = \mathbf{1}_4$ and $(\sigma^{12})^2 = (i\gamma^1\gamma^2)^2 = \mathbf{1}_4$, we get

$$e^{-iseE\sigma^{03}} = \cosh\left(seE\right)\mathbf{1}_4 - i\sinh\left(seE\right)\sigma^{03} \, , \tag{2.19}$$
$$e^{-iseB\sigma^{12}} = \cos\left(seB\right)\mathbf{1}_4 + \sin\left(seB\right)\sigma^{12} \, ,$$

and since $\mathrm{tr}\{\sigma^{03}\} = 0 = \mathrm{tr}\{\sigma^{12}\}$ and $\mathrm{tr}\{\sigma^{03}\sigma^{12}\} = 0$, we have

$$\mathrm{tr}\left[e^{-iseE\sigma^{03}} e^{-iseB\sigma^{12}}\right] = 4\,\cosh\left(seE\right)\,\cos\left(seB\right) \, , \tag{2.20}$$

which appears multiplied by M in its contribution to the integrand in the expression of $2\log S_0[A]$ (see Eq. (2.12)).

Subtracting to this product its $e \to 0$ limit, which gives simply $-i/4\pi^2 s^2$, we finally get for the *one-loop effective action*

(2.21)

$$i \int d^4x\, \mathcal{L}^{(1)}_{eff}(A) := \log S_0[A] =$$

$$\frac{i}{8\pi^2} \int d^4x \int_0^\infty \frac{ds}{s}\, e^{is\left(-m^2 + i\varepsilon\right)}$$

$$\times \left\{ \cosh\left(seE\right)\, \cos\left(seB\right) \frac{eB}{\sin\left(seB\right)}\, \frac{eE}{\sinh\left(seE(1+\imath 0)\right)} - \frac{1}{s^2} \right\},$$

from which it follows that the *one-loop effective Lagrangian* is represented by

(2.22)

$$\mathcal{L}^{(1)}_{eff}(A) := \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s}\, e^{is\left(-m^2 + i\varepsilon\right)}$$

$$\times \left\{ \cosh\left(seE\right)\, \cos\left(seB\right) \frac{eB}{\sin\left(seB\right)}\, \frac{eE}{\sinh\left(seE(1+\imath 0)\right)} - \frac{1}{s^2} \right\}.$$

(Notice that this integral presents a logarithmic divergence at the lower limit, $s = 0$, which shows that a further *renormalization* is needed to properly define the effective Lagrangian.)

Now, we will give this expression a covariant form. Notice that

$$\mathcal{F}^2 + \mathcal{G}^2 = \frac{1}{4}\left(E^2 - B^2\right)^2 + E^2 B^2 = \frac{1}{4}\left(E^2 + B^2\right)^2 . \tag{2.23}$$

Then

$$E^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}\,, \quad B^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}\,, \tag{2.24}$$

or

$$\begin{aligned} a &:= +\sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}} = E\,, \\ b &:= +\sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}} = |B|\,, \end{aligned} \tag{2.25}$$

which replaced in Eq. (2.22) gives

(2.26)
$$\mathcal{L}^{(1)}_{eff}(A) := \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{is\left(-m^2+i\varepsilon\right)} \times \left\{ \frac{e^2 ab\cosh\left(sea\right)\,\cos\left(seb\right)}{\sinh\left(sea\right)\,\sin\left(seb\right)} - \frac{1}{s^2}\right\} =$$
$$= -\frac{1}{8\pi^2}\int_0^\infty \frac{d\tau}{\tau}\, e^{-\tau m^2}\left\{ \frac{e^2 ab\cos\left(\tau ea\right)\,\cosh\left(\tau eb\right)}{\sin\left(\tau ea(1+\imath 0)\right)\,\sinh\left(\tau eb\right)} - \frac{1}{\tau^2}\right\},$$
where, in the last step, we have changed the integration path according to the replacement $s \to -i\tau$.

Notice that the bracket in the last integrand behaves for small τ as

$$-\frac{e^2}{3}\left(a^2-b^2\right) = -\frac{2}{3}e^2\mathcal{F} = \frac{e^2}{6}F_{\mu\nu}F^{\mu\nu}\,, \tag{2.27}$$

which is independent of τ and proportional to the Maxwell Lagrangian.

Therefore, a suitable $\left(O\left(e^2\right)\right)$ counterterm added to the (zero order) Maxwell Lagrangian leads to the following *finite* expression for the electromagnetic effective Lagrangian, which incorporates the first quantum corrections due to the interaction with the quantum Dirac field,

(2.28)
$$\mathcal{L}_{Max}(A) + \mathcal{L}^{(1)}_{eff}(A) = \mathcal{F} - \frac{1}{8\pi^2}\int_0^\infty \frac{d\tau}{\tau}\, e^{-\tau m^2} \times \left\{ \frac{e^2 ab\cos\left(\tau ea\right)\,\cosh\left(\tau eb\right)}{\sin\left(\tau ea(1+\imath 0)\right)\,\sinh\left(\tau eb\right)} - \frac{1}{\tau^2} + \frac{e^2}{3}\left(a^2-b^2\right)\right\}.$$

Notice that these formal manipulations led us to a Lorentz and gauge invariant integrand which is integrable in a neighborhood of $\tau = 0$. It also presents poles at the other zeros of $\sin\left(\tau ea(1+\imath 0)\right)$ in the denominator, placed at $\tau_n = (n\pi/ea)(1-\imath 0)$, with $n = 1, 2, 3, \cdots$).

3. Pair creation

In terms of the electric and magnetic field intensities, the one-loop Heisenberg-Euler effective Lagrangian reads as

(3.1)
$$\mathcal{L}^{(1)}_{HE} := -\frac{1}{8\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t} \times \left[e^2 EB\coth\left(t\frac{eB}{m^2}\right)\cot\left(t\frac{eE}{m^2}(1+i0)\right) - \frac{m^4}{t^2} + \frac{e^2}{3}\left(E^2-B^2\right)\right].$$

This is a well defined expression, since the (simple) poles of the integrand are placed below the integration path:

$$t_n = n\frac{\pi m^2}{eE} - i0\,, \quad n = 1, 2, \cdots . \tag{3.2}$$

Therefore, for $t \simeq t_n$, we have

$$\begin{array}{c} \dfrac{1}{\sin\left(\frac{teE}{m^2} + i0\right)} = \dfrac{(-1)^n}{\sin\left(\frac{teE}{m^2} - n\pi + i0\right)} = \\ = (-1)^n \left\{ \mathrm{PV}\left(\dfrac{1}{\sin\left(\frac{teE}{m^2} - n\pi\right)}\right) - i\,\dfrac{\pi m^2}{eE}\,\delta\left(t - n\dfrac{\pi m^2}{eE}\right)\right\} . \end{array} \tag{3.3}$$

This means that each pole contributes to the imaginary part of the effective Lagrangian with

$$\begin{array}{c} i\,\dfrac{e^2 EB}{8\pi^2 n}\; e^{-n\,\dfrac{\pi m^2}{eE}} \coth\left(\dfrac{n\pi B}{E}\right) \\ \longrightarrow_{B\to 0} \quad i\,\dfrac{e^2E^2}{8\pi^3 n^2}\; e^{-n\,\dfrac{\pi m^2}{eE}} . \end{array} \tag{3.4}$$

Taking into account that the probability of fermionic vacuum persistence in the presence of the electromagnetic background field is given by

$$|\langle \Omega_0\,| S[A] |\,\Omega_0\rangle|^2 = |S_0[A]|^2 = \left| e^{i\int d^4x\, \mathcal{L}^{(1)}_{eff}(A)}\right|^2 = e^{-\int d^4x\,\Gamma(A)}\,, \tag{3.5}$$

where

$$\Gamma(A) := 2\,\mathrm{Im}\left[\mathcal{L}^{(1)}_{HE}(A)\right]\,, \tag{3.6}$$

is the rate of a e^+e^- pair creation per unit volume and unit time. We see that

$$\Gamma(A) = \frac{e^2E^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{e^{-n\frac{\pi m^2}{eE}}}{n^2} = \frac{e^2E^2}{4\pi^3}\, e^{-\dfrac{\pi m^2}{eE}}\left(1 + O\left[e^{-\frac{\pi m^2}{eE}}\right]\right) \tag{3.7}$$

This is only appreciable for $E \approx E_c$, where the *critical value*[1] is $E_c := m^2/e \simeq 1.3 \times 10^{18}$ Volt/meter.

[1] In Dunne's report units, $E_c := m^2c^3/\hbar e$

4. Weak-field asymptotic expansion of the effective Lagrangian

The exponential cut in the integrand in Eq. (3.1) allows one to subtract and add any partial sum of the Laurent series expansion of the product $\coth(tB/E_c)\cot(tE/E_c)$, to generate an asymptotic expansion of the one-loop effective Lagrangian for small values of the electromagnetic background field (compared with E_c).

Let us recall that

$$\coth(z) = \sum_{k=0}^{\infty} \frac{2^{2k}\mathrm{B}_{2k}}{(2k)!}\, z^{2k-1}\,, \quad \cot(z) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}\mathrm{B}_{2k}}{(2k)!}\, z^{2k-1}\,, \tag{4.1}$$

where B_{2k} are the Bernoulli numbers,

$$\mathrm{B}_{2k} = (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}\,\zeta(2k)\,. \tag{4.2}$$

Therefore, we have the asymptotic expansion

$$\begin{aligned}\mathcal{L}^{(1)}_{HE} \asymp -\frac{m^4}{8\pi^2} \sum_{k+n\geq 2} \frac{(-1)^k 2^{2k+2n}\mathrm{B}_{2k}\mathrm{B}_{2n}}{(2k)!(2n)!}\left(\frac{E^{2k}B^{2n}}{E_c^{2k+2n}}\right) \times \\ \times \int_0^\infty t^{2k+2n-3}\, e^{-t} dt = \\ = -\frac{m^4}{8\pi^2} \sum_{k+n\geq 2} \frac{(-1)^k 2^{2k+2n}(2k+2n-3)!\,\mathrm{B}_{2k}\mathrm{B}_{2n}}{(2k)!(2n)!}\left(\frac{a^{2k}b^{2n}}{E_c^{2k+2n}}\right)\,,\end{aligned} \tag{4.3}$$

where we have written the last expression in a Lorentz invariant form, with

$$\begin{aligned} a^2 = \sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}\,, \quad b^2 = \sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F} \\ \Rightarrow \quad a^2-b^2 = 2\mathcal{F}\,, \quad a^2b^2 = \mathcal{G}^2\,. \end{aligned} \tag{4.4}$$

The first terms in this expansion are[2]

(4.5)
$$\mathcal{L}_{HE}^{(1)} \asymp \frac{m^4}{360\,\pi^2 {E_c}^4} \left\{ \left(a^2-b^2\right)^2 + 7a^2b^2 \right\} +$$
$$+ \frac{m^4}{1260\,\pi^2 {E_c}^6} \left(a^2-b^2\right) \left\{ 2\left(a^2-b^2\right)^2 + 13a^2b^2 \right\} + \cdots$$
$$= \frac{m^4}{360\,\pi^2 {E_c}^4} \left\{ 4\mathcal{F}^2 + 7\mathcal{G}^2 \right\} + \frac{m^4}{1260\,\pi^2 {E_c}^6} 2\mathcal{F} \left\{ 8\mathcal{F}^2 + 13\mathcal{G}^2 \right\} + \cdots ,$$

The first term in the RHS is the effective Lagrangian derived by Heisenberg and Euler in 1936. In terms of the two invariants $\mathcal{F}$ and G,

$$\mathcal{L}_M + \mathcal{L}_{HE} = \mathcal{F} + \frac{m^4}{360\,\pi^2 {E_c}^4} \left\{ 4\mathcal{F}^2 + 7\mathcal{G}^2 \right\} , \tag{4.6}$$

It gives the amplitude for light-light scattering at low energy[3] (or large wavelength), a very tiny effect of the order of $\frac{\mathbf{e^4}}{\mathbf{360\pi^2 m^4}} = \frac{\alpha}{\mathbf{90\pi E_c^2}} =: \frac{\varrho}{2}$.

Notice that $\mathcal{L}_{HE}$ contains no derivatives of the electromagnetic field; these corrections would be further suppressed by factors of ω/m (The full process was not solved until 1951[4]).

One can see that the RHS of (4.3) is not a convergent but only a *divergent asymptotic series* by considering the purely magnetic background case ($E=0$). Indeed, we have

$$\mathcal{L}^{(1)} \asymp -\frac{m^4}{8\pi^2} \sum_{n=2}^{\infty} \frac{2^{2n}\,\mathrm{B}_{2n}}{(2n-1)(2n-2)(2n-3)} \left(\frac{B}{E_c}\right)^{2n} . \tag{4.7}$$

where the Bernoulli number has the asymptotic form

$$\mathrm{B}_{2n} \asymp (-1)^{n-1} 4\sqrt{n\pi}\, e^{2n[\log n - \log\pi - 1]} , \tag{4.8}$$

which is alternate in sign and strongly growing with n.

In the purely Electric background, the coefficient has an additional factor $(-1)^n$, and the series is divergent and nonalternating.

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[2]G.V. Dunne, in *From Fields to Strings*, Vol. 1, eds. M. Shifman, A. Vainshtein and J. Wheater (World Scientific, 2005), p. 445.

[3]H. Euler and B. Köckel, *On the scattering of light from light in the Dirac Theory*, Naturwiss **23**, (1935) 246.

[4]R. Karplus and M. Neuman, Phys. Rev. **83**, (1951), 776.

5. Polarization phenomena in electromagnetic backgrounds

We have already mentioned the pair production due to the presence of an electric background of the order of E_c, and the tiny effect of photon - photon scattering mediated by fermions.

There are also dispersive effects on the propagation of an electromagnetic wave produced by the presence of a constant uniform electromagnetic background field.

Basically, the polarization of the fermionic vacuum makes it to act like a birefringent medium with two different indices of refraction, depending on the polarization of the propagating wave.

Let us consider the Heisenberg - Euler correction to the Maxwell Lagrangian,

$$\mathcal{L}_M + \mathcal{L}_{HE} = \mathcal{F} + \frac{m^4}{360\,\pi^2 {E_c}^4}\left\{4\mathcal{F}^2 + 7\mathcal{G}^2\right\}\,, \tag{5.1}$$

with a total electromagnetic field which is the sum of a (nearly) constant and uniform background, $F_{\mu\nu}$, plus the field of an optical wave, $f_{\mu\nu}$, and let us look for the (classical) equation of motion of this fluctuation.

In so doing, it is sufficient to retain the piece quadratic in $f_{\mu\nu}$, since the linear term is a total divergence for a constant background, and the higher order terms (cubic and quartic in $f_{\mu\nu}$) are suppressed by factors of $f_{\mu\nu}/E_c$.

We get[5]

$$\begin{aligned} f_{\mu\nu} f_{\alpha\beta} \Big\{ -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} + \frac{m^4}{360\,\pi^2 {E_c}^4} \big[-2\mathcal{F} g^{\mu\alpha} g^{\nu\beta} + F^{\mu\nu} F^{\alpha\beta} \\ -\frac{7}{4}\mathcal{G}\epsilon^{\mu\nu\alpha\beta} + \frac{7}{4}\widetilde{F}^{\mu\nu}\widetilde{F}^{\alpha\beta}\big]\Big\} = f_{\mu\nu} f_{\alpha\beta} M^{\mu\nu\alpha\beta}\,, \end{aligned} \tag{5.2}$$

where $\widetilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and $M^{\mu\nu\alpha\beta} = M^{\alpha\beta\mu\nu} = -M^{\nu\mu\alpha\beta}$.

Calling

$$\varrho := m^4/(180\,\pi^2 {E_c}^4) = \alpha/(45\pi E_c^2) \tag{5.3}$$

and writing $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, we obtain the modified Maxwell equations by taking the functional derivative

$$\frac{\delta}{\delta a_\sigma}\int d^4x\, f_{\mu\nu} f_{\alpha\beta} M^{\mu\nu\alpha\beta} = 0\,. \tag{5.4}$$

[5]E. Brezin and C. Itzykson, Physical Review **D3**, (1971) 618.

We get

$$\begin{aligned}&\partial_\mu \widetilde{f}^{\mu\nu} = 0\,,\\ &(1+4\varrho\mathcal{F})\,\partial_\mu f^{\mu\nu} - 2\varrho\left[F^{\mu\nu}F^{\alpha\beta} + \frac{7}{4}\widetilde{F}^{\mu\nu}\widetilde{F}^{\alpha\beta}\right]\partial_\mu f_{\alpha\beta} = 0\,.\end{aligned} \tag{5.5}$$

Let us look for a solution of the form $a_\nu(x) = \varepsilon_\nu(k)e^{-ik\cdot x} \Rightarrow f_{\mu\nu} = -i\left[k_\mu\varepsilon_\nu(k) - k_\nu\varepsilon_\mu(k)\right]e^{-ik\cdot x}$, which replaced in the previous equation leads to

$$\begin{aligned}&(1+4\varrho\mathcal{F})\left[k^2\varepsilon^\nu(k) - k^\nu(k\cdot\varepsilon(k)\right] =\\ &= 2\varrho\left[F^{\mu\nu}F^{\alpha\beta} + \frac{7}{4}\widetilde{F}^{\mu\nu}\widetilde{F}^{\alpha\beta}\right]k_\mu\left[k_\alpha\varepsilon_\beta(k) - k_\beta\varepsilon_\alpha(k)\right]\,.\end{aligned} \tag{5.6}$$

This means that the polarization vector must have the form

$$\varepsilon^\nu(k) = \xi_0 k^\nu + \xi_1 k_\mu F^{\mu\nu} + \xi_2 k_\mu \widetilde{F}^{\mu\nu}\,. \tag{5.7}$$

where ξ_0 represents an irrelevant gauge transformation, and can not be determined by the equations.

Replacing in the modified Maxwell equation, we get for the other two parameters

$$\begin{aligned}&(1+4\varrho\mathcal{F})\,k^2\left[k_\mu F^{\mu\nu}\xi_1 + k_\mu\widetilde{F}^{\mu\nu}\xi_2\right] -\\ &-4\varrho\left[F^{\mu\nu}F^{\alpha\beta} + \frac{7}{4}\widetilde{F}^{\mu\nu}\widetilde{F}^{\alpha\beta}\right]k_\mu k_\alpha\left[k^\sigma F_{\sigma\beta}\xi_1 + k^\sigma\widetilde{F}_{\sigma\beta}\xi_2\right] =\\ &= k_\mu F^{\mu\nu}\left\{(1+4\varrho\mathcal{F})\,k^2\xi_1 - 4\varrho k_\alpha F^{\alpha\beta}\left[k^\sigma F_{\sigma\beta}\xi_1 + k^\sigma\widetilde{F}_{\sigma\beta}\xi_2\right]\right\} +\\ &+k_\mu\widetilde{F}^{\mu\nu}\left\{(1+4\varrho\mathcal{F})\,k^2\xi_2 - 7\varrho k_\alpha\widetilde{F}^{\alpha\beta}\left[k^\sigma F_{\sigma\beta}\xi_1 + k^\sigma\widetilde{F}_{\sigma\beta}\xi_2\right]\right\} = 0\,.\end{aligned} \tag{5.8}$$

Considering the independent components of $F^{\mu\nu}$, and taking into account that

$$\begin{aligned}&k^\mu F_{\mu\nu}\widetilde{F}^{\nu\lambda}k_\lambda = k^2\mathcal{G}\\ &k^\mu\widetilde{F}_{\mu\nu}\widetilde{F}^{\nu\lambda}k_\lambda = k^\mu F_{\mu\nu}F^{\nu\lambda}k_\lambda - 2\mathcal{F}\,k^2\,,\end{aligned} \tag{5.9}$$

we see that the coefficients $\xi_{1,2}$ are solutions of the homogeneous system

$$\begin{pmatrix} (1+4\varrho F)\,k^2 + 4\varrho\,k\cdot F\cdot F\cdot k\,, & 4\varrho\,k^2\mathcal{G}\\ 7\varrho\,k^2\mathcal{G}\,, & (1-10\varrho\mathcal{F})\,k^2 + 7\varrho\,k\cdot F\cdot F\cdot k\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0\,, \tag{5.10}$$

where

$$\begin{aligned}&k\cdot F\cdot F\cdot k := k^\mu F_{\mu\nu}F^{\nu\lambda}k_\lambda =\\ &= {k_0}^2\mathbf{E}^2 + \mathbf{k}^2\mathbf{B}^2 - 2k_0\mathbf{k}\cdot\mathbf{E}\times\mathbf{B} - (\mathbf{k}\cdot\mathbf{E})^2 - (\mathbf{k}\cdot\mathbf{B})^2\,.\end{aligned} \tag{5.11}$$

Nontrivial solutions require the determinant of this matrix to vanish.

For simplicity, let us consider the case in which $\mathcal{G} = 0$ ($\mathbf{E} \perp \mathbf{B}$, or pure magnetic or pure electric field). In this case, the *normal modes* are determined by setting the diagonal elements of this matrix equal to zero.

For definiteness, let us consider the case of a pure magnetic background: $F_{12} = B = -F_{21}$ and the other components equal to zero. We get

$$\begin{aligned} &(1-2\varrho B^2)\,k^2 + 4\varrho\left({k_1}^2+{k_2}^2\right)B^2 = 0\,, \quad \text{mode 1},\\ &(1+5\varrho B^2)\,k^2 + 7\varrho\left({k_1}^2+{k_2}^2\right)B^2 = 0\,, \quad \text{mode 2}. \end{aligned} \tag{5.12}$$

which implies for the dispersion relations

$$\begin{cases} {\omega_1}^2 = \mathbf{k}^2 - \left(\dfrac{4\varrho B^2}{1-2\varrho B^2}\right){\mathbf{k}_\perp}^2\,, \\[2ex] {\omega_2}^2 = \mathbf{k}^2 - \left(\dfrac{7\varrho B^2}{1+5\varrho B^2}\right){\mathbf{k}_\perp}^2\,, \end{cases} \tag{5.13}$$

where $k_\perp$ is the component of $\mathbf{k}$ perpendicular to the magnetic field.

Let us introduce the *index of refraction* of each mode through the definitions $\omega := k_0\,,\quad \mathbf{n}(k) := \mathbf{k}/\omega\,,\quad n(k) := |\mathbf{n}(k)|$.

To first order in ϱ, we get for the indices of refraction and polarization vectors

$$\begin{aligned} &{n_1}^2 - 1 = 4\varrho\mathbf{B}^2\sin\theta^2\,,\ \varepsilon_1 = \mathbf{n}\times\mathbf{B} \qquad\qquad (\text{transverse mode})\,,\\ &{n_2}^2 - 1 = 7\varrho\mathbf{B}^2\sin\theta^2\,,\ \varepsilon_2 = \mathbf{B} - \mathbf{n}\,(\mathbf{n}\cdot\mathbf{B}) \quad (\text{parallel mode})\,, \end{aligned} \tag{5.14}$$

where the angle θ is determined by $\cos\theta = \mathbf{k}\cdot\mathbf{B}/|\mathbf{k}||\mathbf{B}|$ and the terms *transverse* and *parallel* refer to the orientation of the polarization vector with respect to the plane determined by the vectors $\mathbf{k}$ and $\mathbf{B}$. We have also chosen the gauge parameter ξ_0 in such a way that ${\varepsilon_{1,2}}^0 = 0$.

Notice that the difference in refraction indices is maximal at $\theta = \pi/2$ ($\mathbf{k} \perp \mathbf{B}$) and vanishing for $\theta = 0$ ($\mathbf{k} \parallel \mathbf{B}$) . Moreover, the polarization vector for the parallel mode is not perpendicular to $\mathbf{k}$: $\mathbf{n}_2 \cdot \varepsilon_2 = (\mathbf{n}_2 \cdot \mathbf{B})(1 - {n_2}^2) \neq 0$.

These two different values of the refraction index imply birefringence. For example, an incident wave with frequency ω, wavelength $\lambda = 2\pi/\omega$ ($c = 1$) and polarization vector $\varepsilon(t) = (\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2)\cos(\omega t)$, which propagates perpendicularly to the uniform magnetic field $\mathbf{B}$ along a distance L, turns into an elliptically polarized wave,

$$\varepsilon(t) = \alpha_1\varepsilon_1\cos[\omega t - 2\pi n_1 L/\lambda] + \alpha_2\varepsilon_2\cos[\omega t - 2\pi n_2 L/\lambda]\,, \tag{5.15}$$

where the phase shift between the two rays is

$$\phi = 2\pi \triangle n \frac{L}{\lambda} \simeq 2\pi \, \frac{3}{2} \varrho \mathbf{B}^2 \, \frac{L}{\lambda} = \frac{\alpha \mathbf{B}^2}{15 {E_c}^2} \, \frac{L}{\lambda} \, . \tag{5.16}$$

On the other hand, the flux of energy follows the direction of the *group velocity*, $\mathbf{v}_G := \partial\omega/\partial\mathbf{k}$, which is given for each mode (to lowest order in ϱ) by

$$\begin{array}{ll} \mathbf{v}_{G,1} = \left(1 - 4\varrho B^2\right) \mathbf{n} + 4\varrho \, (\mathbf{n} \cdot \mathbf{B}) \, \mathbf{B} & \text{(transverse mode)} \, , \\ \mathbf{v}_{G,2} = \left(1 - 7\varrho B^2\right) \mathbf{n} + 7\varrho \, (\mathbf{n} \cdot \mathbf{B}) \, \mathbf{B} & \text{(parallel mode)} \, . \end{array} \tag{5.17}$$

We see that the energy propagates in a direction different from that of its wave vector, as occurs in an anisotropic cristal, except for $\mathbf{n} \perp \mathbf{B}$ or $\mathbf{n} \parallel \mathbf{B}$.

In the case of a purely electric field, it can be seen that the roles of the two modes are interchanged,

$$\left\{ \begin{array}{ll} {n_1}^2 - 1 = 7\varrho \mathbf{E}^2 \sin\theta^2 \, , \ \varepsilon_1 = \mathbf{n} \times \mathbf{E} & \text{(transverse mode)} \, , \\ {n_2}^2 - 1 = 4\varrho \mathbf{E}^2 \sin\theta^2 \, , \ \varepsilon_2 = \mathbf{E} - \mathbf{n} \, (\mathbf{n} \cdot \mathbf{E}) & \text{(parallel mode)} \, , \end{array} \right. \tag{5.18}$$

and similar conclusions are reached.

In the case of crossed fields, $\mathbf{E} \perp \mathbf{B}$, one obtains a similar result to lowest order in ϱ[6]. For example, for low frequency stationary waves, the phase shift produced is expressed in terms of the *intensity* of the standing wave as

$$\phi = \frac{4\alpha I}{15 {E_c}^2} \, \frac{L}{\lambda} \, . \tag{5.19}$$

This is what could be measured in experiments with high-intensity lasers probed by a linearly polarized X-ray pulse.

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[6]T. Heinzl et al. *On the observation of vacuum birefringence*, Optics Communications 267, (2006) 318.